Recursive sequences

1. $a_k - 2a_{k+1} + a_{k+2} \ge 0 \quad \Rightarrow \quad \sum_{k=2i-1}^{2j-1} (a_k - 2a_{k+1} + a_{k+2}) \ge 0 \Rightarrow a_{2i-1} - a_{2i} - a_{2j} + a_{2j+1} \ge 0 \Rightarrow a_{2i-1} + a_{2j+1} \ge a_{2i} + a_{2j}$

$$\Rightarrow \quad \sum_{j=i}^{n} (a_{2i-1} + a_{2j+1}) \ge \sum_{j=i}^{n} (a_{2i} + a_{2j})$$

$$\Rightarrow (n-i+1)a_{2i-1} + (a_{2i+1} + a_{2i+3} + \dots + a_{2n+1}) \ge (n-i+1)a_{2i} + (a_{2i} + a_{2i+2} + \dots + a_{2n}) = (n-i+2)a_{2i} + (a_{2i+2} + \dots + a_{2n})$$

$$\Rightarrow \sum_{i=1}^{n} [(n-i+1)a_{2i-1} + (a_{2i+1} + a_{2i+3} + \dots + a_{2n+1})] \ge \sum_{i=1}^{n} [(n-i+2)a_{2i} + (a_{2i+2} + \dots + a_{2n})]$$

On expanding and grouping terms on L.H.S. and R.H.S., we have

$$n[a_1 + a_3 + \dots + a_{2n+1}] \ge (n+1)[a_2 + a_4 + \dots + a_{2n}] \Rightarrow \frac{a_1 + a_3 + \dots + a_{2n+1}}{n+1} \ge \frac{a_2 + a_4 + \dots + a_{2n}}{n}$$

If the numbers are in A.P., then $a_k - 2a_{k+1} + a_{k+2} = 0$.

By the first part of this question, replacing in the procedure all inequalities by equality, result follows.

Put $a_1 = 1$, $a_k = \sqrt{t^{k-1}}$, then $\begin{cases} a_{2i} = \sqrt{t^{2i-1}} = t^{i-1}\sqrt{t}, & i = 1,2,\dots,n \\ a_{2i-1} = \sqrt{t^{2i-2}} = t^{i-1}, & i = 1,2,\dots,(n+1) \end{cases}$

$a_k - 2a_{k+1} + a_{k+2} = \sqrt{t^{k-1}} - 2\sqrt{t^k} + \sqrt{t^{k+1}} = \sqrt{t^{k-1}}(1 - 2\sqrt{t} + t) = \sqrt{t^{k-1}}(1-\sqrt{t})^2 > 0$, since $0 < t < 1$.

Applying $\frac{a_1 + a_3 + \dots + a_{2n+1}}{n+1} \ge \frac{a_2 + a_4 + \dots + a_{2n}}{n}$,

$$\frac{1 + t + t^2 + \dots + t^n}{n+1} > \frac{\sqrt{t} + t\sqrt{t} + t^2\sqrt{t} + \dots + t^{n-1}\sqrt{t}}{n} = \frac{1 + t + t^2 + \dots + t^{n-1}}{n}\sqrt{t}$$

$$\therefore \quad \frac{1-t^{n+1}}{(n+1)(1-t)} > \frac{1-t^n}{n(1-t)}\sqrt{t} \Rightarrow \quad \frac{1-t^{n+1}}{n+1} > \frac{1-t^n}{n}\sqrt{t}$$

2. We like to use mathematical induction to prove the given proposition.

For $n = 1$, $a_2 = \frac{a_1^{\,2} + 5}{2a_1} = \frac{3^2 + 5}{2 \times 3} = \frac{7}{3} \Rightarrow a_2 - \sqrt{5} = \frac{7}{3} - \sqrt{5} = \frac{14 - \sqrt{5}}{3} = \frac{(3-\sqrt{5})^2}{2 \times 3}$

$$\therefore \quad 0 < a_2 - \sqrt{5} \quad \text{and} \quad a_2 - \sqrt{5} < \frac{(3-\sqrt{5})^2}{(2\sqrt{5})^{2^1 - 1}} = 2\sqrt{5}\left(\frac{3-\sqrt{5}}{2\sqrt{5}}\right)^2 < 6\left(\frac{2}{11}\right)^{2^1}$$

$\therefore$ The proposition is true for $n = 1$.

Assume that the proposition is true for $n = k$. i.e. $0 < a_k - \sqrt{5} < \frac{(3-\sqrt{5})^{2^{k-1}}}{(2\sqrt{5})^{2^{k-1}-1}} < 6\left(\frac{2}{11}\right)^{2^k}$.

We would like to prove that the proposition is true for $n = k + 1$.

$a_{k+1} - \sqrt{5} = \frac{a_k^{\,2} + 5}{2a_k} - \sqrt{5} = \frac{(a_k - \sqrt{5})^2}{2a_k} > 0$, since $a_k > 0$.

$$a_{k+1}-\sqrt{5}=\frac{\left(a_k-\sqrt{5}\right)^2}{2a_k}<\frac{1}{2a_k}\left[\frac{\left(3-\sqrt{5}\right)^{2^{k-1}}}{\left(2\sqrt{5}\right)^{2^{k-1}-1}}\right]^2$$, by inductive hypothesis.

$$=\frac{\sqrt{5}}{a_k}\frac{\left(3-\sqrt{5}\right)^{2^{k+1}}}{\left(2\sqrt{5}\right)^{2^{k+1}-1}}<\frac{\left(3-\sqrt{5}\right)^{2^{k+1}}}{\left(2\sqrt{5}\right)^{2^{k+1}-1}}$$, since by inductive hypothesis $a_k<\sqrt{5}$.

$$\frac{\left(3-\sqrt{5}\right)^{2^{k+1}}}{\left(2\sqrt{5}\right)^{2^{k+1}-1}}=2\sqrt{5}\left(\frac{3-\sqrt{5}}{2\sqrt{5}}\right)^{2^{k+1}}<6\left(\frac{2}{11}\right)^{2^{k+1}}$$

$\therefore$ The proposition is true for $n = k + 1$.

By the principle of mathematical induction, the proposition is true $\forall\ n \in \mathbb{N}$.

3. (i) $y_k = Ay_{k-1} + B$ and $y_{k-1} = Ay_{k-2} + B$

$\Rightarrow$ $y_k - y_{k-1} = A(y_{k-1} - y_{k-2}) = A^2(y_{k-2} - y_{k-3}) = \ldots = A^{k-2}(y_2 - y_1)$

$$\Rightarrow \sum_{i=1}^{k-1}\left(y_{i+1}-y_i\right)=\sum_{i=1}^{k-1}A^{i-1}\left(y_2-y_1\right)\Rightarrow y_k-y_1=\frac{A^{k-1}-1}{A-1}\left(y_2-y_1\right)\Rightarrow y_k-y_1=\frac{A^{k-1}-1}{A-1}\left[Ay_1+B-y_1\right]$$

$$\Rightarrow y_k=\frac{A^{k-1}-1}{A-1}\left[(A-1)y_1+B\right]+y_1=\left(A^{k-1}-1\right)y_1+\frac{A^{k-1}-1}{A-1}B+y_1=A^{k-1}y_1+\frac{A^{k-1}-1}{A-1}B$$

$y_k=A^{k-1}y_1+\dfrac{A^{k-1}-1}{A-1}B$ can also be proved by mathematical induction.

(ii) (a) $x_k = (a + b)\, x_{k-1} - ab\, x_{k-2}$ $(k \ge 2)$

$\Rightarrow$ $x_k - ax_{k-1} = b(x_{k-1} - ax_{k-2}) = b^2(x_{k-2} - ax_{k-3}) = \ldots = b^{k-1}(x_1 - ax_0)$

(b) $x_k - ax_{k-1} = b^{k-1}(x_1 - ax_0)$ $\Rightarrow$ $x_k = ax_{k-1} + b^{k-1}(x_1 - ax_0)$.

Then by (i), Let $A = a$ and $B = b^{k-1}(x_1 - ax_0)$

$$x_k=a^{k-1}x_1+\frac{a^{k-1}-1}{a-1}\left[b^{k-1}\left(x_1-ax_0\right)\right]$$

(iii) Let $x_k = (a + b)\, x_{k-1} - ab\, x_{k-2}$ $(k \ge 2)$, then by comparing coefficients, $a+b=\dfrac{1}{3}, ab=-\dfrac{2}{3}$

Solving we get $(a,b)=\left(-\dfrac{2}{3},\ 1\right)$ or $(a,b)=\left(1,\ -\dfrac{2}{3}\right)$

If $(a,b)=\left(-\dfrac{2}{3},\ 1\right)$, applying (ii) (b), we have

$$x_k=a^{k-1}x_1+\frac{a^{k-1}-1}{a-1}\left[b^{k-1}\left(x_1-ax_0\right)\right]=\left(-\frac{2}{3}\right)^{k-1}x_1+\frac{1-\left(-\frac{2}{3}\right)^{k-1}}{1-\left(-\frac{2}{3}\right)}\left[1^{k-1}\left(x_1-\left(-\frac{2}{3}\right)x_0\right)\right]$$

$$\therefore \lim_{k\to\infty}x_k=\frac{1}{1+\frac{2}{3}}\left(x_1+\frac{2}{3}x_0\right)=\frac{3}{5}\left(x_1+\frac{2}{3}x_0\right)=\frac{3}{5}x_1+\frac{2}{5}x_0$$

If $(a,b)=\left(1,\ -\dfrac{2}{3}\right)$, $\lim\limits_{k\to\infty}x_k$ does not exist and is rejected.

4. Let P(n) be the proposition $\left(\sqrt{3}+1\right)^n = a_n\sqrt{3}+b_n$

For P(1), $\left(\sqrt{3}+1\right)^1 = a_1\sqrt{3}+b_1$, where $a_1 = 1$, $b_1 = 1$. $\therefore$ P(1) is true.

Assume P(k) is true for some $k \in \mathbb{N}$. i.e. $\left(\sqrt{3}+1\right)^k = a_k\sqrt{3}+b_k$ …. (1)

For P(k + 1) .

$$\left(\sqrt{3}+1\right)^{k+1} = \left(\sqrt{3}+1\right)^k\left(\sqrt{3}+1\right) \overset{\text{by (1)}}{=} \left(a_k\sqrt{3}+b_k\right)\left(\sqrt{3}+1\right) = \left(a_k+b_k\right)\sqrt{3}+\left(3a_k+b_k\right) = a_{k+1}\sqrt{3}+b_{k+1}$$

where $a_{k+1} = a_k + b_k$, $b_{k+1} = 3a_k + b_k$ $\therefore$ P(k + 1) is true.

By the principle of mathematical induction, the proposition is true $\forall\, n \in \mathbb{N}$.

(i) $a_{n+2}\sqrt{3}+b_{n+2} = \left(\sqrt{3}+1\right)^{n+2} = \left(\sqrt{3}+1\right)^n\left(\sqrt{3}+1\right)^2 = \left(\sqrt{3}+1\right)^n\left[2\left(\sqrt{3}+1\right)+2\right] = 2\left(\sqrt{3}+1\right)^{n+1}+2\left(\sqrt{3}+1\right)^n$

$= 2\left[a_{n+1}\sqrt{3}+b_{n+1}\right]+2\left[a_n\sqrt{3}+b_n\right] = \left[2\left(a_{n+1}+a_n\right)\right]\sqrt{3}+\left[2\left(b_{n+1}+b_n\right)\right]$

$\therefore$ $a_{n+2} = 2(a^{n+1} + a_n)$, $b_{n+2} = 2(b^{n+1} + b_n)$

(ii) Let P(n) be the proposition $\left(\sqrt{3}-1\right)^n = (-1)^{n-1}\left(a_n\sqrt{3}-b_n\right)$

For P(1), $\left(\sqrt{3}-1\right)^1 = (-1)^{1-1}\left(a_1\sqrt{3}-b_1\right)$, where $a_1 = 1$, $b_1 = 1$. $\therefore$ P(1) is true.

Assume P(k) is true for some $k \in \mathbb{N}$. i.e. $\left(\sqrt{3}-1\right)^k = (-1)^{k-1}\left(a_k\sqrt{3}-b_k\right)$ …. (2)

For P(k + 1) . $\left(\sqrt{3}-1\right)^{k+1} \overset{\text{by (2)}}{=} (-1)^{k-1}\left(a_k\sqrt{3}-b_k\right)\left(\sqrt{3}-1\right) = (-1)^k\left[\left(a_k+b_k\right)\sqrt{3}-\left(3a_k+b_k\right)\right]$

$= (-1)^k\left(a_{k+1}\sqrt{3}-b_{k+1}\right)$, where $a_{k+1} = a_k + b_k$, $b_{k+1} = 3a_k + b_k$.

(iii) $\left(\sqrt{3}+1\right)^n = a_n\sqrt{3}+b_n$ …. (3) $\left(\sqrt{3}-1\right)^n = (-1)^{n-1}\left(a_n\sqrt{3}-b_n\right)$ …. (4)

(3) × (4), $\left(\sqrt{3}+1\right)^n\left(\sqrt{3}-1\right)^n = \left[a_n\sqrt{3}+b_n(-1)^{n-1}\right]\left(a_n\sqrt{3}-b_n\right)$

$\therefore$ $2^n = (-1)^{n-1}(3a_n^2 - b_n^2)$ $\Rightarrow$ $3a_n^2 - b_n^2 = (-1)^{n-1}2^n$.

5. **(i)** Let P(n) be the proposition $a_n \ge n$, $b_n \ge n$ and $a_n^2 - 2b_n^2 = (-1)^n$.

For P(1),
where $a_1 = 1 \ge 1$ $b_1 = 1 \ge 1$. and $a_1^2 - 2b_1^2 = 1^2 - 2(1)^2 = (-1)^1$.
$\therefore$ P(1) is true.

Assume P(k) is true for some $k \in \mathbb{N}$. i.e. $a_n \ge k$, $b_n \ge k$ and $a_k^2 - 2b_k^2 = (-1)^k$. …. (1)

For P(k + 1) . $a_{k+1} = a_k + 2b_k \ge k + 2k = k + 1$, by (1) and $k \ge 1$.

$b_{k+1} = a_k + b_k \ge k + k = k + 1$, by (1) and $k \ge 1$.

Also, $a_{k+1}^2 - 2b_{k+1}^2 = (a_k + 2b_k) - 2(a_k + b_k) = (-1)(a_k^2 - 2b_k^2) = (-1)(-1)^k = (-1)^{k+1}$, by (1)

$\therefore$ P(k + 1) is true.

By the principle of mathematical induction, the proposition is true $\forall\, n \in \mathbb{N}$.

(ii) If n is odd, then $a_n^2 - 2b_n^2 = -1$, $a_n^2 = 2b_n^2 - 1 < 2b_n^2$ and since $b_n^2 > 0$,

$$\frac{a_n^{\ 2}}{b_n^{\ 2}} < 2 \quad \Rightarrow \quad \frac{a_n}{b_n} < \sqrt{2}$$

If n is even, then $a_n^2 - 2b_n^2 = 1$, $a_n^2 = 2b_n^2 + 1 > 2b_n^2$ and since $b_n^2 > 0$,

$$\frac{a_n^{\ 2}}{b_n^{\ 2}} > 2 \quad \Rightarrow \quad \frac{a_n}{b_n} > \sqrt{2}$$

Since $a_n \geq n$, $b_n \geq n$, therefore $a_n \to \infty$, $b_n \to \infty$ as $n \to \infty$.

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n} - \sqrt{2}\right) = \lim_{n\to\infty}\left(\frac{a_n - \sqrt{2}b_n}{b_n}\right) = \lim_{n\to\infty}\frac{a_n^{\ 2} - 2b_n^{\ 2}}{b_n\left(a_n + \sqrt{2}b_n\right)} = \lim_{n\to\infty}\frac{(-1)^n}{b_n\left(a_n + \sqrt{2}b_n\right)} = 0 \quad \Rightarrow \quad \lim_{n\to\infty}\frac{a_n}{b_n} = \sqrt{2}$$

(iii) $\dfrac{a_{n+1}}{b_{n+1}} = \dfrac{a_n + 2b_n}{a_n + b_n}$

$$\therefore \quad \left|\frac{a_{n+1}}{b_{n+1}} - \sqrt{2}\right| = \left|\frac{a_n + 2b_n}{a_n + b_n} - \sqrt{2}\right| = \left|\frac{a_n + 2b_n - \sqrt{2}a_n - \sqrt{2}b_n}{a_n + b_n}\right| = \left|\frac{\left(1-\sqrt{2}\right)\left(a_n - \sqrt{2}b_n\right)}{a_n + b_n}\right| = \left|\frac{1-\sqrt{2}}{\frac{a_n}{b_n} + 1}\right|\left|\frac{a_n}{b_n} - \sqrt{2}\right|$$

$$< \left|\frac{a_n}{b_n} - \sqrt{2}\right| \quad \text{since} \quad \left|1-\sqrt{2}\right| < 1 \quad \text{and} \quad \frac{a_n}{b_n} > 0, \quad \left|\frac{a_n}{b_n} + 1\right| > 1$$

6. $x_n = \sqrt{x_{n-1}y_{n-1}}$, $y_n = \dfrac{x_{n-1} + y_{n-1}}{2} \Rightarrow y_n - x_n = \dfrac{x_{n-1} + y_{n-1}}{2} - \sqrt{x_{n-1}y_{n-1}} = \dfrac{1}{2}\left(\sqrt{x_{n-1}} - \sqrt{x_{n-1}}\right)^2 \geq 0$

$\therefore$ $x_n \leq y_n$ for $n > 1$.